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Finite-size scaling theory for anisotropic percolation models*

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Abstract : Finite-size scaling (FSS) theory for anisotropic percolation models is rarely studied. A simple FSS theory is developed here for anisotropic percolation models considering the cluster size distribution function as a generalized homogeneous function of the system size L and two connectivity lengths $\xi_{\parallel}$ and $\xi_{\perp}$. The scaling theory predicts a new FSS function form for the cluster related quantities in terms of the anisotropic exponent $\theta = \nu_{\parallel}/\nu_{\perp}$, where $\nu_{\parallel}$ and $\nu_{\perp}$ are the connectivity exponents in the longitudinal and transverse directions respectively and a set of new scaling relations are obtained. In the directed percolation (DP) and directed spiral percolation (DSP) models, the clusters generated are anisotropic and they are called anisotropic percolation models. The FSS theory developed here is verified applying to the DP and DSP models.

Keywords : Disordered systems, percolation, anisotropy, finite-size scaling.

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1. Introduction

In disordered systems, geometrical phase transitions [1] occur at the percolation threshold characterized by singularities of cluster related quantities similar to the critical phenomena or second order phase transition in thermodynamic systems [2]. The singularities occurred in phase transitions are generally described by power laws characterized by well-defined critical exponents. The set of critical exponents and the scaling relations among them characterize the universality class of a system. Since most systems are not solvable analytically, numerical methods are employed in order to investigate the critical behavior. At the same time, the numerical results are very often limited by the finite system size. In a finite system, there is rounding and shifting of critical singularities depending on the ratio of correlation length ξ to the linear dimension L of the system. In order to obtain the behavior of the infinite systems, the results of finite systems are generally extrapolated using finite-size scaling (FSS) [3].

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However, there exist very few FSS theories for anisotropic models of statistical physics [3–6]. Usually, the anisotropy considered in the FSS theory is either in the interaction between the constituent particles [4] or in the topology of the system (strip like structure, $L_{\parallel} \times L_{\perp}$) [3]. However, in a geometrical model like percolation, the particles are simply occupied with a probability p in absence of any interaction between them and generally the percolation clusters are generated on topologically isotropic lattices of size $L \times L$. Due to the presence of a global directional constraint, anisotropic percolation clusters are generated in directed percolation (DP) [7] and directed spiral percolation (DSP) [8]. In these models, there are two connectivity lengths $\xi_{\parallel}$ and $\xi_{\perp}$ and they become singular at the percolation threshold with two different critical exponents. For such non-interacting anisotropic models defined on topologically isotropic systems, a simple phenomenological FSS theory is developed here considering the cluster size distribution function as a generalized homogeneous function [2]. The FSS theory developed here predicts a new scaling function from of the cluster related quantities in terms of the anisotropic exponent $\theta = \nu_{\parallel}/\nu_{\perp}$ where $\nu_{\parallel}$ and $\nu_{\perp}$ are the connectivity exponents in the longitudinal and transverse directions respectively. The scaling theory is verified on anisotropic percolation models.

2. Anisotropic percolation models

Anisotropic clusters are generated in two well known percolation models namely DP [7] and DSP [8]. The models are defined on the square lattice of dimension $L \times L$ in 2 dimensions (2D). In DP, a directional field E is present in the model. The direction of applied E field from upper left to the lower right corner of the lattice is considered here. As an effect of the E field, the empty sites to the right and to the bottom of an occupied site are only eligible for occupation. The eligible sites are then occupied with probability p. Consequently, the clusters grow in the diagonal direction along E. In the case of DSP, a crossed rotational field B is also present in addition to the directional field E. In this problem, E field is applied from left to right in the plane of the lattice and B is applied perpendicular to E and out of the plane of the lattice (viewed from top). Due to E field, empty site on the right of an occupied site is eligible for occupation whereas for B field, empty sites in the forward and clockwisely rotational directions are eligible for occupation. As soon as a site is occupied a direction is assigned with it from which it is occupied. The forward direction is the direction from which the present site is occupied. Because of the simultaneous presence of both the E and B fields crossed to each other, a Hall field appears in the system perpendicular to both E and B. As a result, an effective directional constraint $E_{\rm eff}$ acts on the system along the left upper to right lower diagonal of the lattice. The clusters grow along the effective field $E_{\rm eff}$ [8]. A cluster is considered to be a spanning cluster, if either the horizontal or the vertical extension of a cluster becomes equal to the dimension of the lattice L. At the percolation threshold p_c, a spanning cluster appears for the first time in a system ($p_c \approx 0.705489$ for DP [7] and $p_c \approx 0.6550$ for DSP [8] on the square lattice).

Typical large clusters for DP and DSP generated on 256 x 256 square lattice at their respective percolation thresholds are shown in Figure 1. It can be seen that the clusters are anisotropic and rarefied. In order to characterize the cluster's connectivity property, two length scales $\xi_{\parallel}$ along the elongation and $\xi_{\perp}$ perpendicular to the elongation, indicated by the dotted arrows in Figure 1, are required. At $p = p_c$, the connectivity lengths $\xi_{\parallel}$ and $\xi_{\perp}$ diverge as $\xi_{\parallel} \sim |p-p_c|^{-\nu_{\parallel}}$ and $\xi_{\perp} \sim |p-p_c|^{\nu_{\perp}}$, where $\nu_{\parallel}$ and $\nu_{\perp}$ are connectivity exponents. The critical properties of the DSP clusters at $p = p_c$ were found different from that of DP clusters. Accordingly DP and DSP belong to two different universality classes [8].

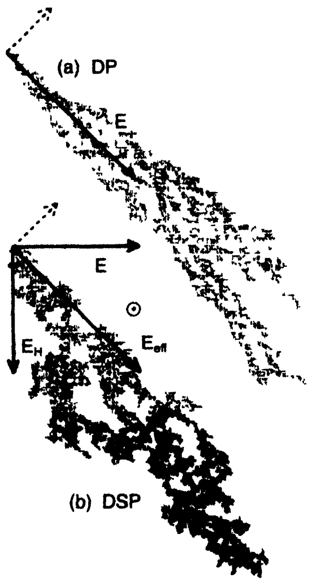

Figure 1. Typical large clusters of (a) directed percolation and (b) directed spiral percolation generated at $p = p_c$ on a square lattice of size $L = 256$. Arrows represent the directional field E and the encircled dot represents the rotational field B. The dotted arrows represent the transverse direction along which $\xi_{\perp}$ is measured.

3. FSS theory

A system is said to be finite if the system size L is less than ξ, the connectivity length. In the case of anisotropic percolation models, there are two connectivity lengths $\xi_{\parallel}$ and $\xi_{\perp}$, where $\xi_{\parallel}$ always greater than $\xi_{\perp}$. The finiteness of the system size is then defined in terms of $\xi_{\parallel}$. A FSS theory is developed here for the anisotropic cluster related quantities which will enable one to estimate the values of the critical exponents for infinite system studying the systems of size less than $\xi_{\parallel}$. According to the theory of critical phenomena, thermodynamic functions become generalized homogeneous functions at the critical point [2]. The cluster size distribution function

$P_s(p,L)$ describing the geometrical quantities here in percolation is then expected to be a generalized homogeneous function at the percolation threshold p_c. In order to develop a FSS theory, the scaling of the order parameter of the percolation transition P_∞, probability to find an occupied site in a spanning cluster, with the system size L is considered first. At the end, the scaling form will be generalized for arbitrary cluster related quantity Q.

In the case of an infinite system, the order parameter P_∞ becomes singular at $p = p_c$ as $P_\infty \sim (p - p_c)^\beta$ with a critical exponent β. In a finite geometry with topologically isotropic dimension $L \times L$, the critical singularity of P_∞ for anisotropic percolation clusters depends not only on $(p - p_c)$ but also on the ratios $\xi_{\|}/L$ and $\xi_\perp/L$. The functional dependence of P_∞ on these parameters is then given by

$$P_\infty(L,p) = G[(p - p_c), \xi_{\|}/L, \xi_\perp/L] \tag{1}$$

For an infinite system, $\xi_{\|}$ is infinitely large at $p = p_c$ and the system properties become independent of the system size L. Accordingly, two parameters $\xi_{\|}/L$ and $\xi_\perp/L$ in P_∞ given in eq. (1) can be reduced to a single parameter $\xi_{\|}/\xi_\perp$. Thus, P_∞ in eq. (1) can be expressed as $P_\infty = \zeta[(p - p_c), \xi_{\|}/\xi_\perp]$. It is now important to know how $\xi_{\|}$ and $\xi_\perp$ scale with the lattice dimension L of the finite system at the percolation threshold p_c. Two new scaling relations for $\xi_{\|}$ and $\xi_\perp$ with L are assumed as,

$$\xi_{\|} \approx L^{\theta_{\|}} \text{ and } \xi_\perp \approx L^{\theta_\perp} \tag{2}$$

where $\theta_{\|}$ and $\theta_\perp$ *are* two new exponents. It is now possible to define P_∞ in terms of the system size L as

$$P_\infty = F\left[(p - p_c), L^{\theta_{\|} - \theta_\perp}\right]. \tag{3}$$

If the scaling function F is a generalized homogeneous function of $(p - p_c)$ and $L^{\theta_{\|} - \theta_\perp}$ then

$$F\left[\lambda^a(p - p_c), \lambda^b L^{\theta_{\|} - \theta_\perp}\right] = \lambda P_\infty \tag{4}$$

where a and b are arbitrary numbers and λ is a parameter [2]. The above relation is valid for any value of λ. For $\lambda = L^{-(\theta_{\|} - \theta_\perp)/b}$, form of P_∞ becomes

$$= L^{A(\theta_{\|} - \theta_\perp)} F\left[(p - p_c)L^{-B(\theta_{\|} - \theta_\perp)}, 1\right] \tag{5}$$

where $A = 1/b$ and $B = a/b$ are two exponents to be determined. However, as $L \to \infty$, the L dependence of P_∞ will vanish. Therefore, $F[z]$ should go as $z^{A/B}$ in the limit $L \to \infty$ when $z = (p - p_c)L^{-B(\theta_{\|} - \theta_\perp)}$. In that case, $P_\infty \approx (p - p_c)^{A/B}$. The order parameter exponent β is then given by $\beta = A/B$.

The size of a cluster is given by the number of occupied sites s in that cluster.

If $R_{\|}$ and $R_{\perp}$ are the radii of gyrations along the two principal axes, it is expected that the cluster size should scale as $s \approx R_{\|}R_{\perp}^{(d_f-1)}$, at $p = p_c$ and it should go as $s \approx R_{\|}R_{\perp}^{(d-1)}$ above p_c where d is the spatial dimension of the lattice and d_f is the fractal dimension of the infinite clusters generated on the same lattice. The percolation probability P_∞ is then given by $P_\infty = R_{\|}R_{\perp}^{(d_f-1)}/R_{\|}R_{\perp}^{(d-1)} = R_{\perp}^{d_f-d}$. Assuming $R_{\perp} \approx \xi_{\perp} \approx L^{\theta_\perp}$ at $p_c, P_\infty \sim L^{\theta_\perp(d_f-d)}$. Also at $p = p_c$, the functional form of P_∞, given in eq. (5), reduces to $P_\infty \sim L^{A(\theta_\| - \theta_\perp)}$. Therefore, exponent A can be obtained in terms of the new exponents $\theta_\|$ and $\theta_\perp$ as $A = (\theta_\perp(d_f - d))/(\theta_\| - \theta_\perp)$. Inserting the value of A in eq. (5) at $p = p_c$ it reduces to $P_\infty = L^{\theta_\perp(d_f-d)}\ F[0] = \xi_\perp^{(d_f-d)} F[0] = \xi_\|^{(d_f-d)\theta_\perp/\theta_\|} F[0]$, where $F[0]$ is a constant. Converting the both sides of the equations in terms of $(p - p_c)$, the following scaling relations can easily be extracted :

$$\beta = \nu_\perp(d - d_f) \text{ and } \beta = \nu_\|(d - d_f)\theta_\perp/\theta_\|. \tag{6}$$

The first one of these relations is the well known hyperscaling relation [1] and the second one is a new scaling relation connecting the exponents $\theta_\|$ and $\theta_\perp$. Using the above scaling relations and eliminating $(d - d_f)$, the values of the exponents A and B can be obtained as

$$A = -\theta_\perp\beta/\left[(\theta_\| - \theta_\perp)\nu_\perp\right] = -\theta_\perp\beta/\left[(\theta_\| - \theta_\perp)\nu_\|\right]$$

and

$$B = A/\beta = -\theta_\perp/\left[(\theta_\| - \theta_\perp)\nu_\perp\right] = -\theta_\|/\left[(\theta_\| - \theta_\perp)\nu_\|\right]. \tag{7}$$

From the expression of A, it can be seen that $\nu_\|/\theta_\| = \nu_\perp/\theta_\perp$. Consequently, one can define the anisotropy exponent

$$\theta = \theta_\|/\theta_\perp = \nu_\|/\nu_\perp \tag{8}$$

as suggested in Ref. [9]. The scaling relation $\beta = \nu_\|(d - d_f)\theta_\perp/\theta_\|$ then reduces to the hyperscaling relation $\beta = \nu_\perp(d - d_f)$. Since the clusters are elongated along the diagonal of the lattice, $\xi_\|$ should be $\approx \sqrt{2}L$ for large clusters and accordingly $\theta_\|$ is expected to be 1. Consequently the anisotropic exponent $\theta \approx 1/\theta_\perp$. The numerical value of $\theta_\perp$ has to be determined in order to verify eq. (8). It is interesting to note that eq. (8) is always valid even if the hyperscaling relations in eq. (6) are not exactly satisfied. It is known that hyperscaling relations are violated in case of DP [10].

Since the values of A and B are now known, the finite-size scaling form of P_∞ can be given as

$$P_\infty(L, p) = L^{-\beta/\theta\nu_\perp} F\left[(p - p_c)\, L^{1/\theta\nu_\perp}\right]. \tag{9}$$

A finite size scaling relation is thus obtained in terms of the transverse connectivity length exponent $\nu_\perp$ and the anisotropy exponent θ.

Thus, in general, a quantity Q which diverges as $Q \sim |p - p_c|^{-q}$ in the infinite system as $p \to p_c$, should obey a finite-size scaling law given by

$$Q(L, p) = L^{q/\theta\nu_\perp} F\left[(p - p_c) L^{1/\theta\nu_\perp}\right]. \tag{10}$$

It should be noticed that the finite size scaling relation obtained here describes the scaling of cluster related quantities in the transverse direction. This is obtained without varying the system size in the transverse direction. The FSS relation in eq. (10) is different from that of Binder and Wang [5] obtained for Ising system on the rectangular geometry $L_{\|} \times L_{\perp}$. Therefore, the FSS relation obtained here in terms of anisotropy exponent θ and connectivity exponent $\nu_\perp$ in the transverse direction is a new scaling relation. The anisotropy exponent θ (eq. 8) is measured below for both DP and DSP clusters and the transverse FSS theory developed here is verified for both the models.

4. Verification of FSS Theory

In order to verify the proposed FSS theory, simulations of DP and DSP models are performed on the square lattice of sizes L = 128 to 2048 in multiple of 2. Average has been taken over 5×10^4 large finite clusters. First, the exponents $\theta_\perp$ and $\theta_{\|}$ which describe the dependence of connectivity lengths with the system size L are determined. Since, clusters are grown following single cluster growth algorithm, the connectivity lengths are given by $\xi_{\|}^2 = 2\sum_s' R_{\|}^2 s P_s(p, L) / \sum_s' P_s(p, L)$ and $\xi_{\|}^2 = 2\sum_s' R_\perp^2 s P_s(p, L) / \sum_s' P_s(p, L)$, where $R_{\|}$ and $R_\perp$ are radii of gyration with respect to two principal axes of the cluster. $R_{\|}$ and $R_\perp$ are estimated from the eigenvalues of the moment of inertia tensor, a 2×2 matrix here. The cluster size distribution function $P_s(p,L)$ is defined as N_s/N_{tot}, where N_s is the number of s-sited clusters out of N_{tot} clusters generated on a given system size. $\xi_\perp$ is measured for various system sizes L at $p = p_c$ for both DP and DSP clusters and plotted against the system size L in Figure 2. The exponent $\theta_\perp$ is found different for DP and DSP : $\theta_\perp = 0.64 \pm 0.01$ for DP and $\theta_\perp = 0.83 \pm 0.01$ for DSP. The exponent $\theta_{\|}$ is also measured and found ≈ 1 for both the models as expected. Assuming $\theta_{\|} = 1$, one should have $\theta_\perp = \nu_\perp/\nu_{\|}$. Since for DP, $\nu_\perp = 1.0972 \pm 0.0006$ and $\nu_{\|} = 1.7334 \pm 0.001$ the expected value of $\theta_\perp$ is ≈ 0.633. Similarly for DSP, the expected value of $\theta_\perp$ is ≈ 0.84 since $\nu_\perp = 1.12 \pm 0.03$ and $\nu_{\|} = 1.33 \pm 0.01$. The measured values are close to the expected values for both DP and DSP. The scaling relation in eq. (8) is then verified under error bar. Thus the anisotropy exponent $\theta = 1/\theta_\perp \approx 1.56$ for DP and ≈ 1.20 for DSP. It can also be noticed that the magnitude of $\xi_\perp$ is different for DP and DSP. $\xi_\perp$ is larger for the DSP clusters in comparison to that of DP clusters. Consequently the DSP clusters are less anisotropic than DP clusters.

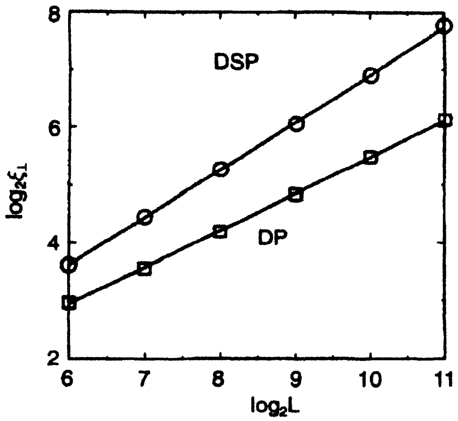

Figure 2. Plot of $\xi_\perp$ *versus* system size L for the DP (□) and the DSP (o) clusters at respective percolation thresholds p_c. From the slopes, value of $\theta_\perp$ is obtained as 0.64 ± 0.01 for DP and 0.83 ± 0.01 for DSP clusters respectively.

Next, the average cluster size χ is measured for different system size L at $p = p_c$. In single cluster growth algorithm, the average cluster size is defined as $\chi = \sum_s' P_s(p, L)$, where $P_s(p,L)$ is the cluster size distribution function. In an infinite system, χ diverges as : $\chi \sim |p - p_c|^{-\gamma}$, γ is a critical exponent. According to the FSS theory, it should behave as $\chi(L) \sim L^{\gamma/\theta\nu_\perp}$ at $p = p_c$. In Figure 3, the average cluster size χ is plotted against the system size L for both DP and DSP. The cluster size χ follows a power law with the system size L. The obtained slopes are 1.31 ± 0.01 for DP and 1.38 ± 0.01 for DSP. The expected value of the ratio of the exponents is $\gamma/\theta\nu_\perp \approx 1.33$ for DP, where $\gamma = 2.2772 \pm 0.0003$. For DSP $\gamma = 1.85 \pm 0.01$ and the expected ratio of the exponents is $\gamma/\theta\nu_\perp \approx 1.37$. It can be seen that the measured values are in agreement with that of the expected values within error bars. It confirms that the cluster properties follow the proposed anisotropic finite size scaling theory.

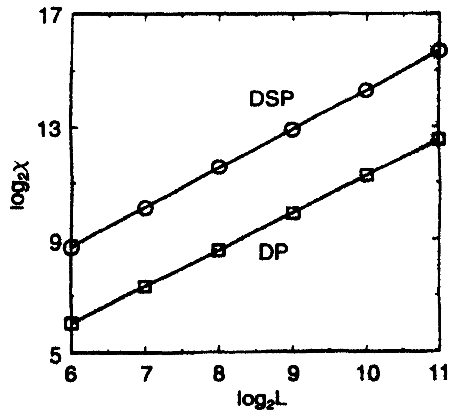

Figure 3. Plot of average cluster size χ *versus* system size L for the DP (□) and DSP (o) clusters at p_c. From the slopes, the ratio of $\gamma/\theta\nu_\perp$ is obtained as 1.31 ± 0.01 and 1.38 ± 0.01 for DP and DSP respectively.

Finally, the FSS function form has been verified. The average cluster size is given as $\chi(L, p) = L^{\gamma/\theta\nu_\perp} F\left[(p - p_c)L^{1/\theta\nu_\perp}\right]$ for different system size L. In Figure 4, the scaled average size $\chi/L^{\gamma/\theta\nu_\perp}$ is plotted against the scaled variable $z = (p - p_c)L^{1/\theta\nu_\perp}$. It can be seen that a reasonable data collapse is obtained. The tail of the scaling function $F(z)$ shows a power law behavior in both the models with two different scaling exponents, approximately 2.23 for DP and 1.86 for DSP. Note that, these exponents are close to the respective cluster size critical exponents γ for infinite systems as expected. Once again it confirms that DP and DSP follow anisotropic finite size scaling and belong to two different universality classes.

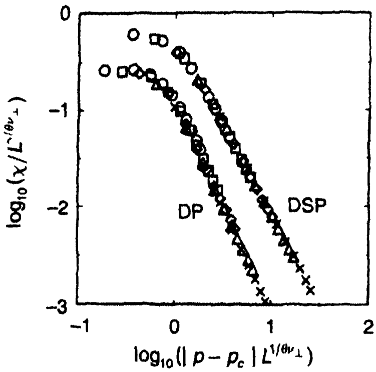

Figure 4. Plot of scaled average cluster size $\chi/L^{\gamma/\theta\nu_\perp}$ *versus* scaled variable $(p - p_c)L^{1/\theta\nu_\perp}$ for DP and DSP clusters. The data plotted correspond to different system sizes of L = 128 (o), 256 (□), 512 (◇), 1024 (△) and 2048 (×). The $|p - p_c|$ values here are 0.01 to 0.10 in the interval of 0.01.

5. Conclusion

A new finite size scaling theory is proposed here for anisotropic percolation models like DP and DSP. In this theory the cluster size distribution is assumed to be a generalized homogeneous function of $(p - p_c)$ and the ratios of the connectivity lengths $\xi_\parallel$ and $\xi_\perp$ to the system size L. The scaling function form of the cluster properties is obtained in terms of the anisotropic exponent $\theta = \nu_\parallel/\nu_\perp$, where $\nu_\parallel$ and $\nu_\perp$ are the connectivity exponents in the longitudinal and transverse directions respectively and a set of new scaling relations are obtained. The proposed scaling relations as well as the scaling function form are verified *via* numerical simulation. It could be considered as a simplest possible anisotropic finite size scaling theory for anisotropic percolation models.

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